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ON THE NON-LINEAR THEORY OF THE PLANE
TRAVELING WAVE TUBE

BY
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TECHNICAL NOTE NO 1 PREPARED UNDER
US AIR FORCE CONTRACT AF 61 (052) – 594

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RESEARCH REPORT NO 37

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ABSTRACT

This paper deals with the non-linear theory of a plane
TWT model. An infinitely wide electron beam interacts with
an infinitely wide homogeneous slow wave structure whereby
the difficulties associated with radial boundary conditions
are avoided. The non-linear wave equation is solved to the
third order accuracy by the use of the method of successive
approximations. The one-valued velocity assumption limits
the analysis to the range well below the saturation level.
Within this range, however, the results describe such ef-
fects as the dependence of fundamental frequency gain and
output phase on drive power, the production of the first
two harmonics, etc. Furthermore, it is shown that, by the
use of suitable approximations, one can write the non-
linear plane TWT equation in a form very similar to the
exact non-linear plane klystron equation. From this point
on one can deal with the plane TWT using the methods which
have already been successfully applied in the plane klys-
tron case.

I. DESCRIPTION OF THE PLANE TWT MODEL

The principles of space charge wave devices, such as klystrons, double stream amplifiers, velocity step amplifiers, resistive medium amplifiers, microwave vacuum diodes, etc., are most conveniently analyzed using plane models with infinitely wide beams. Such models lend themselves for comparatively convenient studies of the basic effects in the various devices. The price one has to pay is the neglect of the important phenomena produced by the fringe fields in real tubes. Nevertheless, since the TWT is an extremely difficult object to study, especially in the non-linear region, it should be of some interest to develop a non-linear theory for the plane TWT model.

The linearized analysis of the plane TWT has been published elsewhere [1]. The plane TWT consists of an infinitely wide confined electron beam moving with a constant dc velocity v_o in an anisotropically conducting homogeneous medium (Fig. 1). The medium represents the slow wave structure. It has to be anisotropic since an isotropic homogeneous medium would support plane TEM waves. The TWT mechanism, however, depends on the presence of an electric field component in the direction of the electron and wave motion
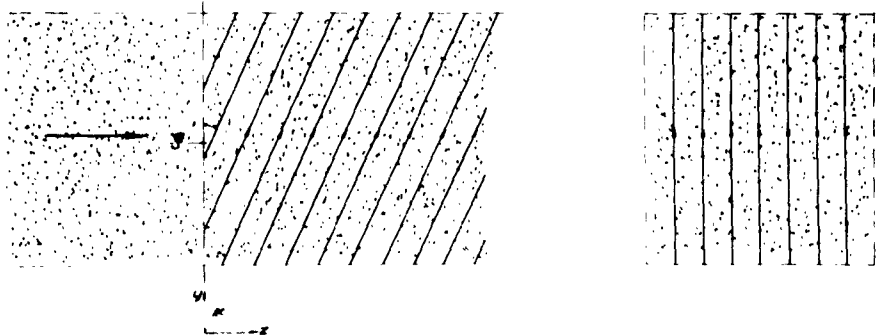
Fig. 1. The infinitely wide electron beam moving in the infinitely wide wire structure.

and thus requires a TM wave. A simple TM slow wave medium
can be thought of as consisting of closely placed straight
parallel wires filling the whole three-dimensional space.
The wires are parallel to the y-z plane while they make an
angle ζ with the x-y plane. The angle ζ corresponds direct-
ly to the pitch angle of a helical slow wave structure.
Mathematically we will take the presence of the wires into
account by letting the conductivity of the whole space be
infinity in the wire direction and zero in all other direc-
tions. The wire-medium is, of course, supposed to be trans-
parent with respect to the beam electrons, which move along
the positive z-direction, The beam is confined by a very
strong dc magnetic field.

   In the absence of the electron beam the medium supports
a plane TM wave with field components E_y, E_z, H_x propaga-
ting in the positive (or negative) z-direction with a phase
velocity $v_{ph} = c \sin\zeta$ (c = velocity of light). The constant
phase and amplitude planes are assumed to be perpendicular
to the z-axis, i.e. $\partial/\partial x = 0 = \partial/\partial y$. In the presence of
the beam interaction will take place and the four charac-
teristic TWT wave are easily found from Maxwell's equations.

II. THE NON-LINEAR WAVE EQUATION

Let us define a quantity $z_1(z,t)$, the displacement of an electron from the position z_0 it would have if ac forces were not applied. If z is the actual position of the electron, one has the relation $z = z_0 + z_1$. Similarly we denote the electron velocity by $v = v_0 + v_1$, the convection current density by $i = i_0 + i_1$ and the electronic charge density by $\rho = \rho_0 + \rho_1$. The quantities indexed by o refer to undisturbed conditions while the quantities indexed by 1 denote the change produced by the signal.

From the relations

$$v_1 = \frac{dz_1}{dt} \tag{1}$$

$$i_1 = \rho_0 v_1 + v_0 \rho_1 + v_1 \rho_1 \tag{2}$$

and

$$\frac{\partial i_1}{\partial z} = - \frac{\partial \rho_1}{\partial t} \tag{3}$$

one easily obtains [2] the expressions

$$i_1 = \rho_0 \frac{\partial z_1}{\partial t} \tag{4}$$

and

$$\rho_1 = - \rho_0 \frac{\partial z_1}{\partial z} \tag{5}$$

The non-relativistic equation of motion is

$$\frac{d^2 z_1}{dt^2} = - \frac{e}{m} E_z \tag{6}$$

Relations (4) and (6) will now be used in Maxwell's equations in order to obtain a non-linear wave equation for z_1. Maxwell's equations yield for the TM case with $\partial/\partial x = 0 = \partial/\partial y$.

$$\left.\begin{array}{l} \dfrac{\partial E_y}{\partial z} = \mu_o \dfrac{\partial H_x}{\partial t} \\[3mm] \dfrac{\partial H_x}{\partial z} = I_y + \varepsilon_o \dfrac{\partial E_y}{\partial t} \\[3mm] 0 = I_z + \varepsilon_o \dfrac{\partial E_z}{\partial t} \end{array}\right\} \qquad (7)$$

where μ_o and ε_o are the permeability and permittivity respectively of free space. I_y and I_z are the current density components

$$\left.\begin{array}{l} I_y = I \cos\zeta \\[3mm] I_z = I \sin\zeta + i_1 \end{array}\right\} \qquad (8)$$

where I is the conduction current density along the direction of infinite conductivity. Since the electric field must be perpendicular to the direction of infinite conductivity, we also have the relation

$$E_y \cos\zeta + E_z \sin\zeta = 0 \qquad (9)$$

By the use of Equations (4), (6), (8) and (9) in (7) one easily deduces the desired wave equation

$$\boxed{\frac{\partial^2}{\partial z^2} \frac{d^2 z_1}{dt^2} - \frac{1}{v_{ph}^2} \frac{\partial^2}{\partial t^2} \frac{d^2 z_1}{dt^2} - \frac{\omega_p^2}{v_{ph}^2} \cos^2\zeta \frac{\partial^2 z_1}{\partial t^2} = 0} \qquad (10)$$

where

$$v_{ph} = (\epsilon_o \mu_o)^{-\frac{1}{2}} \sin\zeta = c \sin\zeta$$

is the phase velocity of the TM wave in the absence of the electron beam and

$$\omega_p = (- \frac{\bullet \rho_o}{m\epsilon_o})^{\frac{1}{2}}$$

is the angular plasma frequency.

Equation (10) is the exact wave equation in the system under consideration. It is non-linear on account of the quantity $d^2 z_1/dt^2$ as we will see in what follows. It should be pointed out that Equation (10) is valid only in such regions where overtaking does not occur since it has been assumed that the total convection current density in a plane z is given by Relation (4). If overtaking occurs then z_1 is not any more a one valued function of z and Relation (4) does not give the total convection current density which we need in Maxwell's first equation.

With $d/dt = \partial/\partial t + (v_o + v_1)\partial/\partial z$ we obtain after some elementary calculations the following non-linear exact expression for $d^2 z_1/dt^2$, viz.

$$\frac{d^2 z_1}{dt^2} = \frac{\ddot{z}_1}{(1 - \frac{\partial z_1}{\partial z})} + \frac{\frac{\partial}{\partial z}(\dot{z}_1)^2}{(1 - \frac{\partial z_1}{\partial z})^2} + \frac{(\dot{z}_1)^2 \frac{\partial^2 z_1}{\partial z^2}}{(1 - \frac{\partial z_1}{\partial z})^3} \tag{11}$$

where

$$\dot{z}_1 \equiv (\frac{\partial}{\partial t} + v_o \frac{\partial}{\partial z}) z_1$$

and

$$\ddot{z}_1 \equiv (\frac{\partial^2}{\partial t^2} + 2v_o \frac{\partial^2}{\partial t \partial z} + v_o^2 \frac{\partial^2}{\partial z^2}) z_1$$

Thus the dot denotes the linearized time derivative. Note, however, that Equation (11) is an exact expression for the electron acceleration. In the case of the linearized theory (see Chapter III) one would approximate Equation (11) by the expression $d^2 z_1/dt^2 \approx \ddot{z}_1$.

The excess velocity v_1 is related to z_1 by the exact expression

$$v_1 = \frac{dz_1}{dt} = \frac{\dot{z}_1}{(1 - \frac{\partial z_1}{\partial z})} \qquad . \qquad (12)$$

Upon the insertion of Equation (11) into Equation (10) one would obtain the complete non-linear partial differential equation for z_1. Obviously this equation, in spite of the simplicity of the plane TWT model, is an extremely complicated one. There is no hope to find a proper exact solution. Nevertheless, it is interesting to note, that the function $e^{j(\omega t - \gamma z_0)}$, where ω and γ are constants [*], is an exact solution to Equation (10). This solution, however, is physically meaningless since its real and imaginary parts taken separately are not exact solutions to Equation (10). Nevertheless, we will show in Chapter VI that certain properties of the plane TWT can be studied by the use of this solution.

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[*] The constant γ is the same as in Equation (16).

III. THE LINEARIZED SOLUTION

If we linearize our wave equation (10), the result is

$$Da_1 = 0 \qquad\qquad (13)$$

where D denotes the operator

$$D = \left(\frac{\partial^2}{\partial t^2} - \frac{1}{v_{ph}^2}\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial t^2} + 2v_0\frac{\partial^2}{\partial t\,\partial z} + v_0^2\frac{\partial^2}{\partial z^2}\right)$$

$$\qquad\qquad (14)$$

$$\quad - \frac{\omega_p^2}{v_{ph}^2}\cos^2\xi\,\frac{\partial^2}{\partial t^2}$$

while a_1 is the linearized electron displacement z_1. An amplifying solution to Equation (13) is

$$a_1 = Ae^{\alpha z}\cos(\omega t - \beta z) = A\,\mathrm{Re}\,(e^{j(\omega t - \gamma z)}) \qquad (15)$$

where $\beta = \mathrm{Re}\,\gamma$ and $\alpha = \mathrm{Im}\,\gamma$ (α is supposed to be positive). The complex propagation constant γ obeys the dispersion equation

$$(\gamma^2 - \gamma_0^2)(\gamma - \beta_e)^2 + \beta_p^2\gamma_0^2\cos^2\xi = 0 \qquad (16)$$

which is obtained by inserting Relation (15) into Equation (13). The notations are:

$\gamma_0 = \omega/v_{ph}$, the propagation constant in the absence of the beam

$\beta_e = \omega/v_0$, the mathematical beam propagation constant

and

$\beta_p = \omega_p/v_0$, the plasma propagation constant.

Equation (16) determines the real quantities α and β.
For the amplifying wave α is positive. We are interested
in cases where v_o is very close to v_{ph} and where $\omega_p^2 \ll \omega^2$.
Under these usual conditions γ will be very close to β_e and
γ_o. Thus Equation (16) can be approximated by

$$(\gamma-\beta_e)^2(\gamma-\gamma_o) + \frac{1}{2}\beta_p^2\beta_e\cos^2\zeta \approx 0 \quad . \tag{17}$$

Using the same assumptions in Pierce's [3] Equation
(7.10) we would get

$$(\gamma-\beta_e)^2(\gamma-\gamma_o) + (\beta_e C)^3(1 - 4Q\frac{\gamma-\gamma_o}{\beta_e}) \approx 0 \tag{18}$$

where C and Q are the well-known parameters defined by
Pierce.

We observe that the two Equations (17, 18) are equi-
valent if

$$C^3 = \frac{1}{2}\frac{\omega_p^2}{\omega^2}\cos^2\zeta \tag{19}$$

and if QC is a small number ($4QC \ll 1$). The latter require-
ment shows that the plane TWT bears a close resemblance to
the small Q helical TWT (strong coupling between the beam
and the helix, beam radius equals the helix radius [4, 5]).

Returning to Equation (17) we introduce the notations
originally used by Pierce

$$\gamma = \beta_e + \beta_e C(jx-y)$$

$$\gamma_o = \beta_e + \beta_e Cb$$

and rewrite Equation (17) in the form

$$(jx-y)^2(jx-y-b) + 1 = 0 \quad . \tag{20}$$

If $|b| \ll |jx-y|$ (i.e. operating point near velocity

synchronism, the region of maximum amplification) we have
the approximate solution

$$x \simeq \frac{\sqrt{3}}{2}$$

$$y \simeq -\frac{1}{2} - \frac{1}{3} b \quad .$$

The parameter x is associated with the exponential
growth of the wave (positive x corresponds to amplification)
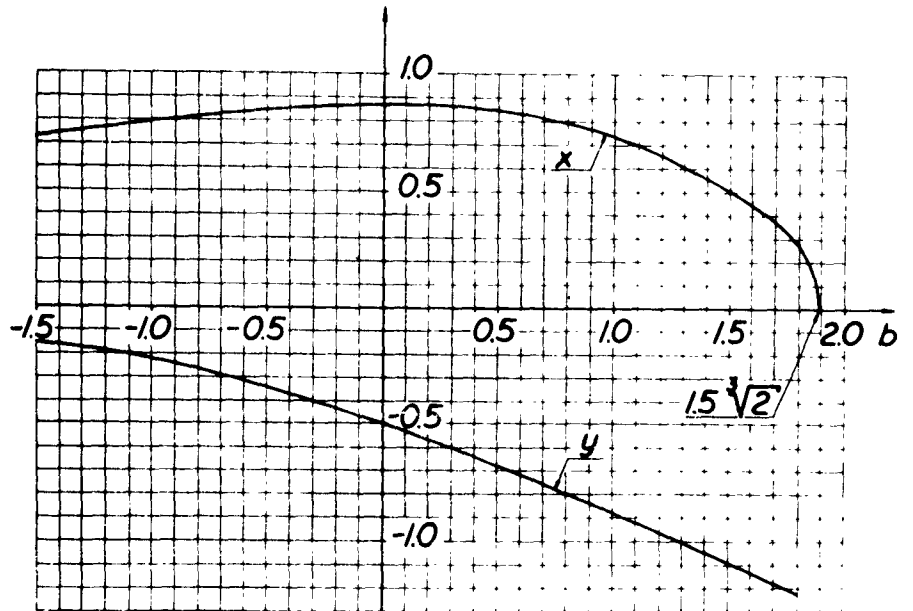whereas y is related to the propagation constant of the wave.

Fig. 2. Depicting the gain parameter x and the phase para-
meter y versus the electron velocity parameter b for the
plane TWT. The parameters x, y and b are those of Pierce [3].

In this connection it is interesting to compare the
exponential linearized amplification of the principal con-
tinuous interaction type amplifiers. The electronic ampli-

fication is given by the factor $e^{2Im(\gamma)z}$. In Fig. 3
$Im(\gamma)/\beta_e$ is depicted versus ω_p^2/ω^2 [*] for the following
amplifiers, all adjusted for maximum amplification.

> A - the plane TWT [1a]
>
> B - the sheath helix TWT, the radius of the solid
>   electron beam equals that of the helix [1b, 4, 5]
>
> C - the plane resistive medium tube [6] and the plane
>   double stream tube [3]. In the latter case
> $\omega_p^2 = 2\omega_{p_1}^2$ where ω_{p_1} is the angular plasma frequen-
>   cy of one beam alone.

A— wire structure tube
B— helical-type TW-tube
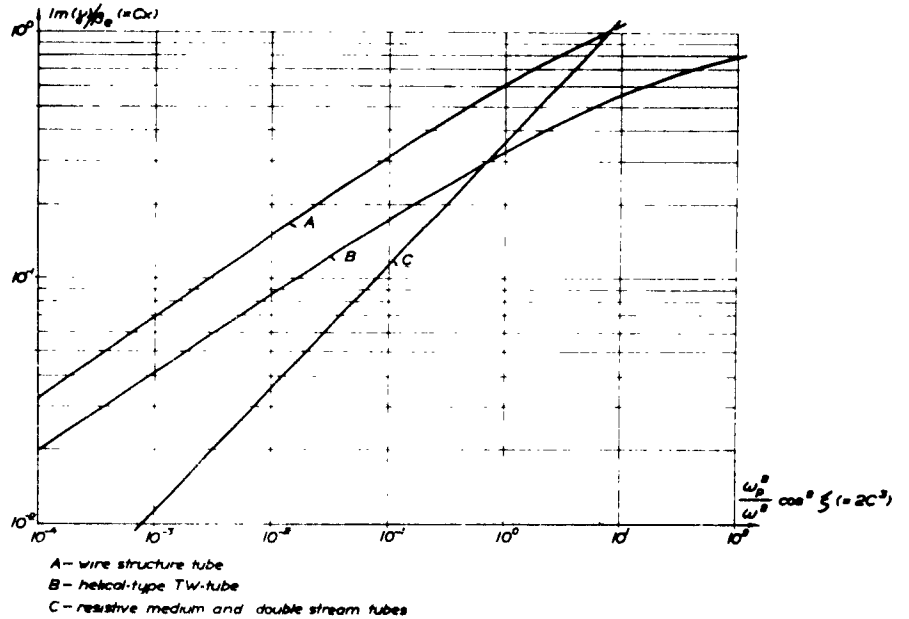C— resistive medium and double stream tubes

Fig. 3. Normalized exponential gain parameter versus nor-
malized dc current density parameter for the principal
continuous interaction type plane wave amplifiers.

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[*] It is assumed, as usual, that $\cos^2\zeta \simeq 1$.

IV. THE NON-LINEAR SOLUTION,
METHOD OF SUCCESSIVE APPROXIMATIONS

It is well known that a set of three linear forward waves will be excited at the input of a TWT. One of these waves will be an amplifying wave. This is the wave described by Relation (15). When the waves have traveled far enough the amplifying wave will dominate over the two other waves. Thus at the distance where the propagation becomes non-linear we need to consider only the amplifying wave and the non-linear effects associated with this particular wave.

We want to make use of the method of successive approximations [7] in order to find a third order solution to Equation (10). This means that we think of the solution in terms of a power series in $Ae^{\alpha z}$, the amplitude of the linearized displacement [see Relation (15)], and want to find the first three terms in the series. The third term contains a non-linear wave of the fundamental frequency and allows us to study the low-level non-linear effects at the frequency ω. The third term also describes the excitation of the third harmonic 3ω while the second harmonic 2ω and a dc correction constitute the second term in the power series.

The total electronic charge density can never be positive, thus $\rho_0 + \rho_1 \leq 0$. By the use of Relation (4) one easily concludes now that $\partial z_1 / \partial z < 1$. Assuming $|\partial z_1 / \partial z| < 1$ we expand Relation (11) as follows

$$\frac{d^2 z_1}{dt^2} = [1 + \frac{\partial z_1}{\partial z} + (\frac{\partial z_1}{\partial z})^2] \ddot{z}_1$$

$$+ [1 + 2 \frac{\partial z_1}{\partial z}] \frac{\partial (\dot{z}_1)^2}{\partial z} + \frac{\partial^2 z_1}{\partial z^2} (\dot{z}_1)^2 \tag{22}$$

Relation (22) includes the non-linear terms to the third

order. Expanding Relation (12) in the same manner yields

$$v_1 = \frac{dz_1}{dt} \simeq [1 + \frac{\partial z_1}{\partial z} + (\frac{\partial z_1}{\partial z})^2]\dot{z}_1 \tag{23}$$

We will now assume that z_1 can be written in the form

$$z_1 = a_1 + a_2 + a_3 \tag{24}$$

where $a_1 \propto Ae^{\alpha z}$, $a_2 \propto (Ae^{\alpha z})^2$ and $a_3 \propto (Ae^{\alpha z})^3$ are the three first terms of the power series. The linear term a_1 is already expressed in Relation (15). By the use of Equations (24) and (22) in Equation (10) one obtains upon equating terms of equal powers in $Ae^{\alpha z}$ the following three equations:

$$Da_1 = 0 \tag{25}$$

$$Da_2 = - (\frac{\partial^2}{\partial z^2} - \frac{1}{v_{ph}^2} \frac{\partial^2}{\partial t^2}) [\frac{\partial a_1}{\partial z} \ddot{a}_1 + \frac{\partial}{\partial z}(\dot{a}_1)^2] \tag{26}$$

$$Da_3 = - (\frac{\partial^2}{\partial z^2} - \frac{1}{v_{ph}^2} \frac{\partial^2}{\partial t^2}) [\frac{\partial a_1}{\partial z} \ddot{a}_2 + \frac{\partial a_2}{\partial z} \ddot{a}_1$$

$$+ 2 \frac{\partial}{\partial z}(\dot{a}_1 \dot{a}_2) + (\frac{\partial a_1}{\partial z})^2 \ddot{a}_1 \tag{27}$$

$$+ 2 \frac{\partial a_1}{\partial z} \frac{\partial}{\partial z}(\dot{a}_1)^2 + \frac{\partial^2 a_1}{\partial z^2}(\dot{a}_1)^2]$$

The solution to Equation (25) is known. Thus Equation (26) is a linear non-homogeneous wave equation. Its particular integral, a_2, can be found by standard methods. Once a_1 and a_2 are known, one can attack Equation (27) and find its particular integral, a_3. The results, written in complex notations, are

$$\beta_e a_1 = (A\beta_e e^{\alpha z})\ e^{j(\omega t - \beta z)} \tag{28}$$

$$\beta_e a_2 = (A\beta_e e^{\alpha z})^2 \left\{ \frac{1}{2} j [1 + C(-y+jx)] e^{2j(\omega t - \beta z)} \right.$$
$$\left. + [\frac{y}{4x} - \frac{3C}{8x}(x^2+y^2)] \right\} \tag{29}$$

$$\beta_e a_3 \simeq (A\beta_e e^{\alpha z})^3 [-\frac{3}{8} e^{3j(\omega t - \beta z)}$$
$$- \frac{(y+b-3jx)(x^2+y^2-2jxy)}{1-(y+b-3jx)(y-3jx)^2} e^{j(\omega t - \beta z)}] \tag{30}$$

Relations (28) and (29) are exact solutions to Equations (25) and (26) respectively, whereas Relation (30) is deduced with the same approximations as Equation (17) earlier.

Remembering that $E_z = -\frac{m}{e}\frac{d^2 z_1}{dt^2}$ we can write $E_z = E_{z_1} + E_{z_2} + E_{z_3}$ where

$$\frac{E_{z_1}}{2\beta_e V_o C^2 (-y+jx)^2} = (A\beta_e e^{\alpha z})\ e^{j(\omega t - \beta z)} \tag{31}$$

$$\frac{E_{z_2}}{2\beta_e V_o C^2 (-y+jx)^2} = (A\beta_e e^{\alpha z})^2\ \frac{1}{2} j[1+C(-y+jx)]\ e^{2j(\omega t - \beta z)} \tag{32}$$

$$\frac{E_{z_3}}{2\beta_e V_o C^2 (-y+jx)^2} \simeq (A\beta_e e^{\alpha z})^3 [-\frac{3}{8} e^{3j(\omega t - \beta z)}$$
$$- \frac{(x^2+y^2-2jxy)}{(-y+jx)^2}\ \frac{e^{j(\omega t - \beta z)}}{1-(y+b-3jx)(y-3jx)^2}] \ . \tag{33}$$

$V_o \ (= \frac{mv_o^2}{2e})$ denotes the beam voltage.

We also express the velocity $v_1 = dz_1/dt$ by writing $v_1 = v_{11} + v_{12} + v_{13}$ where

$$\frac{v_{11}}{v_o} \frac{1}{C(-y+jx)} = -j(A\beta_e e^{\alpha z}) e^{j(\omega t-\beta z)} \tag{34}$$

$$\frac{v_{12}}{v_o} \frac{1}{C(-y+jx)} = (A\beta_e e^{\alpha z})^2$$

$$\cdot \left\{ \frac{1}{2}[1+C(-y+jx)] e^{2j(\omega t-\beta z)} + \frac{C}{4}(y+jx) \right\} \tag{35}$$

$$\frac{v_{13}}{v_o} \frac{1}{C(-y+jx)} \simeq (A\beta_e e^{\alpha z})^3 \left\{ \frac{3}{8}j e^{3j(\omega t-\beta z)} \right.$$

$$+ j[\frac{(y+b-3jx)(x^2+y^2-2jxy)}{1-(y+b-3jx)(y-3jx)^2} \frac{(-y+3jx)}{(-y+jx)} \tag{36}$$

$$\left. - \frac{1}{4} \frac{(3y+jx)}{(-y+jx)}] e^{j(\omega t-\beta z)} \right\} .$$

From the relation $i_1/i_o = \partial z_1/\partial(v_o t)$ we finally express the ac current density, writing $i_1 = i_{11}+i_{12}+i_{13}$ where

$$\frac{i_{11}}{i_o} = (A\beta_e e^{\alpha z}) j e^{j(\omega t-\beta z)} , \tag{37}$$

$$\frac{i_{12}}{i_o} = -(A\beta_e e^{\alpha z})^2 [1+C(-y+jx)] e^{2j(\omega t-\beta z)} , \tag{38}$$

$$\frac{i_{13}}{i_o} \simeq (A\beta_e e^{\alpha z})^3 [-\frac{9}{8}j e^{3j(\omega t-\beta z)}$$

$$- j \frac{(y+b-3jx)(x^2+y^2-2jxy)}{1-(y+b-3jx)(y-3jx)^2} e^{j(\omega t-\beta z)}] . \tag{39}$$

In the synchronous case (b = 0) these results can be written in the form

$$\beta_e z_1 \simeq R\cos Z + R^2[\tfrac{1}{2}\sin 2Z - \tfrac{1}{4\sqrt{3}}]$$

$$\qquad\qquad (40)$$

$$- R^3[\tfrac{3}{8}\cos 3Z + \tfrac{7}{72}\cos Z + \tfrac{7\sqrt{3}}{72}\sin Z]$$

$$\frac{E_z}{2\beta_e v_o c^2} \simeq R\sin(Z-30^\circ) - \tfrac{1}{2}R^2\cos(2Z-30^\circ)$$

$$- R^3[\tfrac{3}{8}\sin(3Z-30^\circ)+\tfrac{1}{18}\sin(Z-30^\circ)-\tfrac{\sqrt{3}}{36}\cos(Z-30^\circ)]$$

$$\qquad\qquad (41)$$

$$\frac{v_1}{v_o c} \simeq R\cos(Z+30^\circ) + R^2[\tfrac{1}{2}\sin(2Z+30^\circ)-\tfrac{C}{4}]$$

$$- R^3[\tfrac{3}{8}\cos(3Z+30^\circ)+\tfrac{7}{72}\cos(Z+30^\circ)+\tfrac{3\sqrt{3}}{72}\sin(Z+30^\circ)]$$

$$\qquad\qquad (42)$$

and

$$\frac{i_1}{i_o} \simeq R\sin Z - R^2\cos 2Z$$

$$\qquad\qquad (43)$$

$$- R^3[\tfrac{9}{8}\sin 3Z + \tfrac{7}{72}\sin Z - \tfrac{7\sqrt{3}}{72}\cos Z]$$

where $R = A\beta_e e^{\alpha z}$ and $Z = \beta z - \omega t$.

In Fig. 4 Relations (40, 41, 42, 43) have been plotted versus Z, the phase, for three values of the amplitude parameter R. Observe that the abscissa, Z, in these graphs can be interpreted as the normalized axial distance, βz, if time is kept constant and if R varies negligibly within a distance of one axial wavelength (this implies $2\pi\alpha/\beta \ll 1$).
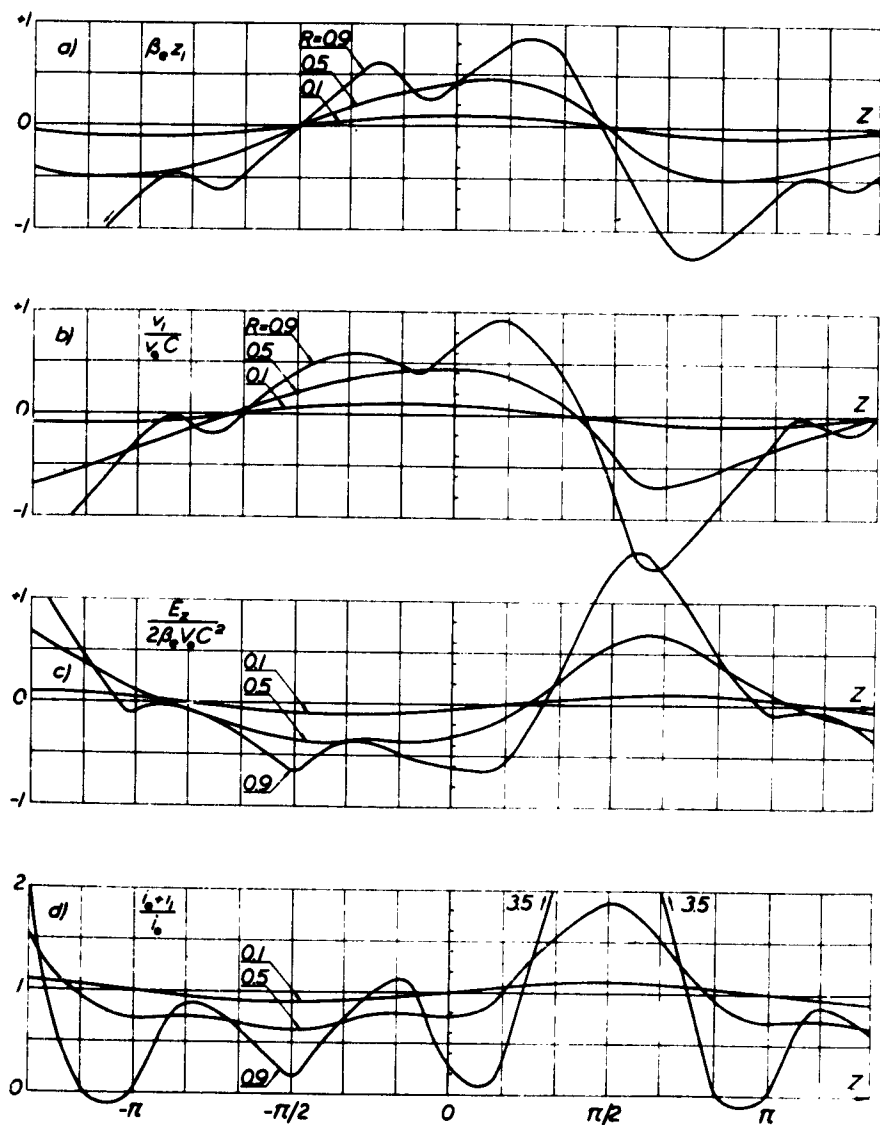
Fig. 4. Depicting a) the displacement (z_1), b) the excess velocity (v_1), c) the axial electric field strength (E_z) and d) the total electron current density $(i = i_0 + i_1)$ as a function of the phase $(Z = \beta z - \omega t)$ for three values of the amplitude parameter $(R = A\beta_e e^{\alpha z})$.

At the small amplitude R = 0.1 all curves in Fig. 4
are practically sinusoidal. When R = 0.5 the non-lineari-
ties clearly manifest their presence and the curves become
non-sinusoidal. In the case of R = 0.9 one can see from
Fig. 4 or from Relations (40, 41, 42, 43) that the second
and third order terms are comparable to the first order
terms. Thus the third order theory will not describe the
situation very accurately for such large R. Note, for in-
stance, that the total current density (Fig. 4d) becomes
negative for some values of Z when R = 0.9. This would
correspond to a negative electron number density, which,
of course, is unacceptable. However, extrapolating Fig. 4a
to even larger R, one would expect the situation
$\partial(\beta_e z_1)/\partial Z = -\infty$, which is the electron overtaking condi-
tion, to occur at Z slightly less than $\frac{\pi}{2}$. This is in good
agreement with the experimental findings of C.C.Cutler [8].
Fig. 4c indicates that the Z-range, where the electric
field is decelerating (E_z positive), narrows when R in-
creases. In this useful Z-range, according to Fig. 4b,
most electrons travel slower than the undisturbed electrons
(i.e. v_1 is negative). Observe finally (Fig. 4d) that, for
large R, a strong electron bunch is built up somewhat to
the left of the middle of the decelerating range (around
$Z \simeq \pi/2$).

V. NONLINEARITIES OF THE AMPLITUDE AND THE PHASE

The purpose of this chapter is to study how the low level nonlinearities manifest themselves in the amplitude and phase of the fundamental frequency waves.

Denoting the fundamental frequency component of E_z by E_{z_ω}, one obtains from Equations (31) and (33)

$$\frac{E_{z_\omega}}{2\beta_e V_o c^2 R(-y+jx)^2} = [1-R^2 f_E(x,y,b)]e^{j(\omega t-\beta z)} \qquad (44)$$

where

$$f_E(x,y,b) = \frac{(x^2+y^2-2jxy)}{(-y+jx)^2[1-(y+b-3jx)(y-3jx)^2]} \qquad (45)$$

Similarly Equations (37) and (39) yield

$$\frac{i_{1_\omega}}{i_o Rj} = [1-R^2 f_i(x,y,b)]e^{j(\omega t-\beta z)} \qquad (46)$$

where

$$f_i(x,y,b) = \frac{(y+b-3jx)(x^2+y^2-2jxy)}{1-(y+b-3jx)(y-3jx)^2} \qquad (47)$$

Since the theory presented here is based on the method of successive approximations, we must assume $|R^2 f_{E,i}| \ll 1$. We can therefore write for the right hand sides of Equations (44) and (46)

$$(1-R^2 f_{E,i})e^{j(\omega t-\beta z)} \simeq (1-S_{E,i})e^{j(\omega t-\beta z-\Phi_{E,i})} \qquad (48)$$

where

$$S_{E,i} = R^2 \operatorname{Re}(f_{E,i}) \quad \text{and} \quad \Phi_{E,i} = R^2 \operatorname{Im}(f_{E,i}) \qquad (49,50)$$

Obviously S_E and S_i are the lowest order non-linear amplitude corrections to the electric field and electron current density waves respectively at the fundamental frequency. Similarly Φ_E and Φ_i are the lowest order non-linear phase corrections to these waves. Observe that positive S indicates that the wave amplitude is less than the linearized theory would predict. Positive Φ indicates that the phase is delayed.

Thus we can write

$$E_{z_\omega} = E_{z_1}(1-S_E)e^{-j\Phi_E} \tag{51}$$

$$i_{1_\omega} = i_{11}(1-S_i)e^{-j\Phi_i} \tag{52}$$

where the linearized waves E_{z_1} and i_{11} are given by Equations (31) and (37).

For the electromagnetic power flow P_{em_ω} one finds

$$\frac{P_{em_\omega}}{P_{em_1}} = \left|\frac{E_{z_\omega}}{E_{z_1}}\right|^2 \simeq 1-2S_E$$

where P_{em_1} is the electromag⁻ ⁻c power flow associated with the linearized solution. .nus, with $S_P = 2S_E$, one has

$$P_{em_\omega} = P_{em_1}(1-S_P) \tag{53}$$

It is convenient, before we proceed, to express the parameter $R(= A\beta_e e^{\alpha z})$ in terms of physically more meaningful quantities. For this purpose we make use of the non-relativistic kinetic power theorem [9]

$$P_{k_1} = - \frac{m v_o}{2e} \, \mathrm{Re}(v_{11} i_{11}^*) = - \frac{m i_o}{2e} \, \mathrm{Re}(\dot{a}_1 \frac{\partial a_1^*}{\partial t}) \qquad (54)^{*)}$$

where P_{k_1} is the time average kinetic power flow associated with the linearized waves. Now, the total linearized power flow, which is the sum of P_{k_1} and the corresponding electromagnetic power flow P_{em_1}, must vanish for obvious reasons (see Reference [5] p. 27). Thus $P_{em_1} = -P_{k_1}$. Denoting the undisturbed dc kinetic power flow by P_o one easily obtains by the use of Equations (34) and (37) in (54) that

$$\eta \equiv \frac{P_{em_1}}{P_o} = - C y R^2 \qquad (55)$$

where η may be called the efficiency at the distance z. Note that η is the efficiency corresponding to the linearized theory.

By the use of Equation (55) in (49) and (50) one can now write

$$\frac{S_{E,i}}{\eta/C} = - \frac{\mathrm{Re}(f_{E,i})}{y} \qquad (56)$$

and

$$\frac{\Phi_{E,i}}{\eta/C} = - \frac{\mathrm{Im}(f_{E,i})}{y} \qquad (57)$$

---

*) The complete second order expression for the time average kinetic power flow is $P_k = P_{k_1} + P_{k_2}$ where P_{k_1} is given by Eq. (54) and where $P_{k_2} = - \frac{m i_o}{e}(\frac{1}{4}v_{11}v_{11}^* + v_o v_{12dc})$. The notation v_{12dc} stands for the dc part of the velocity expressed by Equation (35). The term P_{k_2}, although of the same order as P_{k_1}, is associated with the lowest order non-linear waves [9]. In the case treated here one easily proves by the use of Equations (34) and (35) that $P_{k_2} = 0$. This is as expected with respect to the fact that there is no electromagnetic power present, which could balance out a non-zero P_{k_2}.

Equations (56) and (57) have been plotted in Fig.s 5 and 6 respectively. Note from Fig. 5 that the theory predicts negative S_p-values for sufficiently high beam voltages (b > 1). The conclusions are that in this interesting beam velocity range the gain of a TWT initially increases when the drive increases and that, when b $\cong$ 1, the gain is independent of the drive within the accuracy of the third order theory. From the practical standpoint this means that, according to the present theory, there exists an optimum beam voltage which should be used in situations where a linear relationship between the input and output powers up to high drive levels is essential. This optimum beam voltage is higher (b $\approx$ 1) than the voltage corresponding to maximum gain (b $=$ 0). Thus some gain has to be sacrificed in order to increase amplitude linearity.

Fig. 7. shows some experimental results obtained with an RCA 4010 traveling wave tube. The output power was
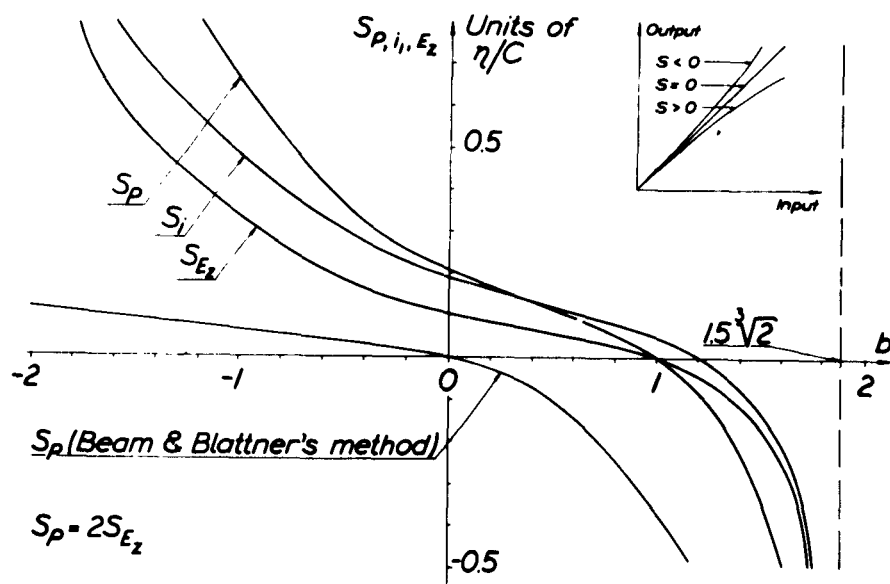

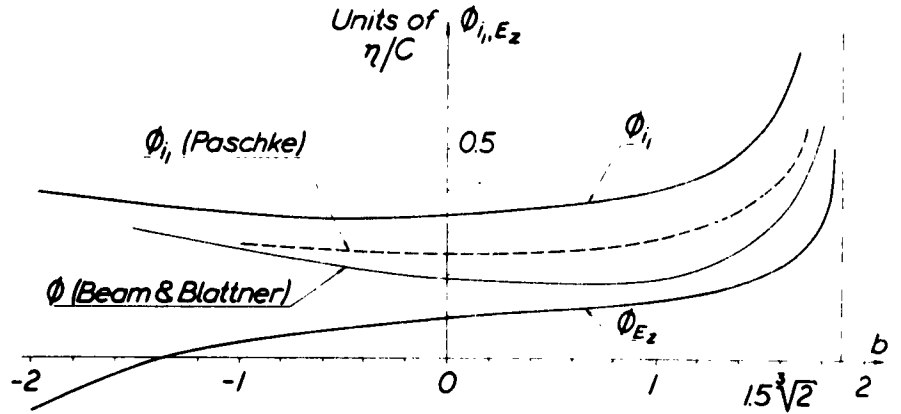Fig. 5. Depicting the amplitude nonlinearity parameters $S_{E,i,P}$ versus the electron velocity parameter b.

Fig. 6. Depicting the phase delay of the ac electric field, Φ_E, and the phase delay of the ac current density, Φ_i, versus the electron velocity parameter b.
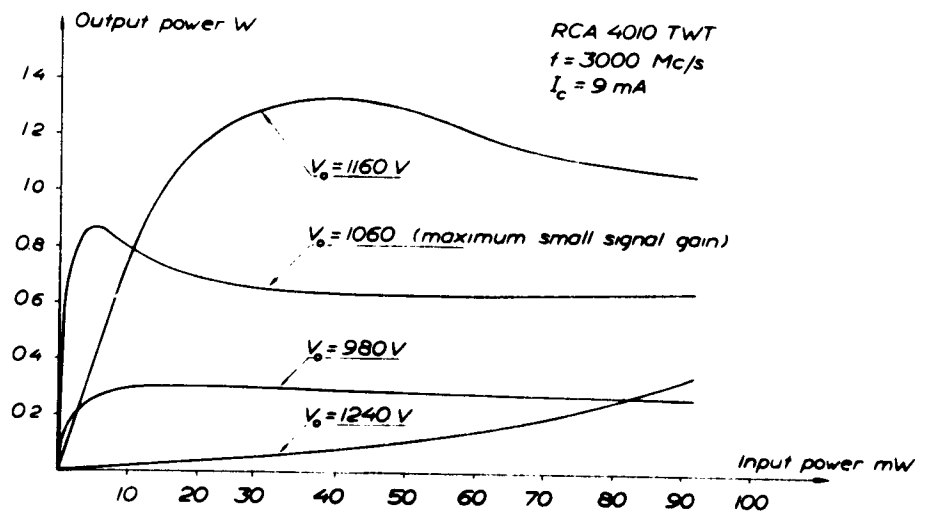

Fig. 7. The output power versus the input power of an RCA 4010 traveling wave tube for some different values of beam/helix voltage.

measured as a function of the input power with the beam/helix
voltage as a parameter. The experimental results are in
good qualitative agreement with the aforesaid theoretical
predictions. The beam voltage V_o = 1060V corresponds to
maximum small signal gain. At this voltage saturation is
produced by 5mW input power. When V_o = 1160V one has app-
roximately $S_p \simeq 0$ and the initial phase of the curve is
essentially linear. Gain is lower but the efficiency is
higher than in the case V_o = 1060V. When V_o = 1240V we
clearly have a negative S_p which was predicted by the
theory. At the low voltage V_o = 980V we see that S_p is
large and positive, also this in good agreement with the
theory.

Beam and Blattner [10] have considered non-linear
effects in traveling wave tubes in an early paper. Their
approach is based on the assumption that the energy de-
livered by the beam to the electromagnetic wave can be
interpreted in terms of a continuous decrease of the aver-
age electron velocity v_o as the beam travels along. From
a linearized theory one easily calculates the energy lost
by the beam. It can then be calculated how v_o, i.e. b,
varies with distance. Assuming that Equation (20) remains
valid, it is now possible to calculate the continuous
change in the complex propagation constant, i.e. the varia-
tion of x and y. A WKB type phase integral has to be used,
of course, in order to describe the waves. The results are,
in our earlier notations,

$$\frac{S_p}{\eta/c} \approx \frac{dx/db}{2x} \tag{58*)}$$

and

$$\frac{\Phi_{E,i}}{\eta/c} \approx \frac{1+dy/db}{4x} \tag{59*)}$$

---

*) The quantities dx/db and dy/db can be obtained from Fig. 2.

Equation (58) is also plotted in Fig. 5 while Equation (59) is plotted in Fig. 6. It is interesting to note that this rather unsophisticated method gives results with the principal features in acceptable qualitative agreement with our results. Beam's and Blattner's method, which has been used by them for phase studies, is, of course, not entirely correct. However, its physical basis is simple and the approach may be helpful in developing a physical understanding of the phenomena involved.

Fig. 6 shows the phase delay versus b. Observe that, for sufficiently slow beam velocities (b < -1.4), the electric field is accelerated rather than delayed by non-linear phenomena (Φ_E negative).

Attention should be called to the fact that the various non-linear waves (E_z, i_1, v_1, etc.) are described by different axial wave functions rather than by one common wave function, which is the case with the linearized waves. Thus the amplitude corrections S are different for the E_z- and i_1-waves and the corresponding phase delays Φ also differ from each other. (See Fig.s 5 and 6.) Nevertheless, the method of Beam and Blattner naturally gives one common wave function also in the non-linear case. The parameters S and Φ, obtained from their theory, consequently apply to all wave quantities involved (E_z, i_1, etc.).

The dashed curve in Fig. 6 has been computed by Paschke [11], who under certain simplifying assumptions has developed a non-linear successive approximation theory for the ordinary delay line type TWT (i.e. a radially finite device). The curve, for QC = 0, has been taken from Paschke's Fig. 3. It describes the phase delay of the fundamental frequency ac current wave (Φ_1). The qualitative agreement between Paschke's and our results is obviously good. However, Paschke assumes without any further discussion that his Φ_1 is identical to the phase delay of the signal in the output terminal. Now, the output signal is the amplified electromagnetic wave whose phase delay,

as we have shown, is not the same as that of the electronic ac current wave.

The phase delay at the signal output is, of course, also depending on the matching conditions at the output. However, it is not possible to use the plane TWT model for detailed studies of the phenomena at the output in a realistic manner [1b]. Nevertheless, with reference to the Φ_E-curve in Fig. 6, it might be possible to reduce the non-linear output phase delay at the expense of gain if the beam/helix voltage is made less than that corresponding to maximum gain.

VI. THE NON-LINEAR SOLUTION,
METHOD OF DOMINANT TERMS

In this Chapter we intend to show that it is possible
to develop a very elegant method for investigations of the
dynamic non-linearities in the plane TWT. The method yields
the dominant part of all harmonic frequency waves (fre-
quencies $n\omega$, $n = 1, 2, 3, 4 ...$) propagating in the system.
The integer n will not be limited to 3, which was the case
in Chapters IV and V. In addition the method offers a possi-
bility to deal with the plane TWT by the use of the theory
of the plane klystron tube [12, 13].

It would have involved very cumbersome calculations if
we had tried to develop the successive approximation approach
to higher order than the third. However, it is easy to see
from the procedure used in connection with the treatment of
Equations (25), (26) and (27), that the dominating term of
the frequency ω appears in the first order solution. The
higher order solutions contain smaller terms of the frequen-
cy ω. These terms describe saturation and non-linear phase
shift effects. The dominating term of the frequency 2ω
appears in the second order solution, the dominating term
of the frequency 3ω in the third order solution, etc. We
ignore now the saturation and phase shift terms and attempt
to find the dominating terms for all harmonic frequencies.

We start from the original wave equation (10) which
is written

$$(\frac{\partial}{\partial z^2} - \frac{1}{v_{ph}^2}\frac{\partial^2}{\partial t^2})\frac{d^2 z_1}{dt^2} - \frac{\omega_p^2}{v_{ph}^2}\cos^2\zeta\frac{\partial^2 z_1}{\partial t^2} = 0 \tag{60}$$

Next we assume that z_1 can be written in the form

$$z_1 \simeq \sum_{n=1}^{\infty} A_n e^{jn(\omega t-\gamma z)} \tag{61}$$

where complex notations have been used for simplicity. The terms in the series are supposed to be the dominant terms just discussed. Saturation and phase shift terms are neglected. The quantity γ is, as before, a solution of Eq. (16).

The acceleration can be written in the form

$$\frac{d^2 z_1}{dt^2} = \ddot{z}_1 + \frac{d^2 g}{dt^2} \tag{62}$$

where the double dot, of course, is the linearized second order time derivative, $\partial^2/\partial t^2 + 2v_0 \partial/\partial z + v_0^2 \partial^2/\partial z^2$. The quantity $d^2 g/dt^2$, actually defined by Eq. (62), takes care of that part of the acceleration which is not included in the term $\ddot{z}_1$.

If the series (61) is a solution to Equation (60), then $d^2 g/dt^2$ must be of the form

$$\frac{d^2 g}{dt^2} \simeq \sum_{n=2}^{\infty} B_n e^{jn(\omega t - \gamma z)} \tag{63}$$

The frequency ω does not appear in the series (63) since the term $\ddot{z}_1$ in Equation (62) naturally contains the dominant term of the frequency ω (which is given by the first order or linearized solution) and the other terms of frequency ω are ignored.

By the use of Equation (62) in (60) one obtains

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{v_{ph}^2}\frac{\partial^2}{\partial t^2}\right)\ddot{z}_1 - \frac{\omega_p^2}{v_{ph}^2}\cos^2\xi\frac{\partial^2 z_1}{\partial t^2} + \left(\frac{\partial^2}{\partial z^2} - \frac{1}{v_{ph}^2}\frac{\partial^2}{\partial t^2}\right)\frac{d^2 g}{dt^2} = 0 \tag{64}$$

Now, the term $A_1 e^{j(\omega t - \gamma z)}$ is our linearized solution, i.e. it is a solution to Equation (64) with the term containing $d^2 g/dt^2$ ignored. With this in mind one easily proves by the use of Equation (61) in the first two terms

of Equation (64) that[*)]

$$(\frac{\partial^2}{\partial z^2} - \frac{1}{v_{ph}^2}\frac{\partial^2}{\partial t^2})\ddot{z}_1 - \frac{\omega_p^2}{v_{ph}^2}\cos^2\xi\frac{\partial^2 z_1}{\partial t^2} =$$

$$= (\frac{\partial^2}{\partial z^2} - \frac{1}{v_{ph}^2}\frac{\partial^2}{\partial t^2})\sum_{n=1}^{\infty}(1-\frac{1}{n^2})(-n^2)(\omega-\gamma v_0)^2 A_n e^{jn(\omega t-\gamma z)} =$$

$$= (\frac{\partial^2}{\partial z^2} - \frac{1}{v_{ph}^2}\frac{\partial^2}{\partial t^2})[\ddot{z}_1 + (\omega-\gamma v_0)^2 z_1]$$

(65)

Note that it has been possible to eliminate the term

$$\frac{\omega_p^2}{v_{ph}^2}\cos^2\xi\frac{\partial^2 z_1}{\partial t^2}$$

Furthermore, Relation (65) can be used in Equation (64), which after dropping the operator $\partial^2/\partial z^2 - (1/v_{ph}^2)\partial/\partial t^2$ yields

$$\ddot{z}_1 + \frac{d^2 f}{dt^2} + (\omega-\gamma v_0)z_1 = 0$$

or, with Equation (62),

$$\boxed{\frac{d^2 z_1}{dt^2} + (\omega-\gamma v_0)^2 z_1 = 0}$$

(66)

---

*) In order to obtain this result one has to observe that

$$\frac{\omega_p^2}{v_{ph}^2}\cos^2\xi\frac{\partial^2}{\partial t^2}S_n = (\frac{\partial^2}{\partial z^2} - \frac{1}{v_{ph}^2}\frac{\partial^2}{\partial t^2})\frac{1}{n^2}\ddot{S}_n \text{ where } S_n = e^{jn(\omega t-\gamma z)}$$

What has now been shown is that the dominant terms at the various frequencies in the solution to the complicated fourth order equation (60) are identical to those in the solution to the much simpler second order equation (66).

It is interesting to note that both Equation (60) and (66) possess the exact solution

$$z_1 = Ae^{j(\omega t - \gamma z_o)}$$

However, it was pointed out in Chapter II that the real and imaginary parts taken separately are not exact solutions to Equation (60). As far as Equation (66) is concerned, the real and imaginary parts are, indeed, exact solutions. This means that a reasonably good non-linear solution to our problem is

$$z_1 = Ae^{\alpha z_o}\cos(\omega t - \beta z_o) =$$
$$= Ae^{\alpha z}e^{-\alpha z_1}\cos(\omega t - \beta z + \beta z_1)$$

(67)

Since $|\alpha z_1| <<< 1$ we can also write

$$\beta_e z_1 = R\cos(\omega t - \beta z_o)$$

(68)

or with $Z_o = \beta z_o - \omega t$

$$\beta_e z_1 = R\cos Z_o$$

(69)

Equation (69) is plotted in Fig. 8a for R = 0.1, R = 0.5 and R = 1.0. If $\beta_e \approx \beta$ and if $2\pi\alpha/\beta \ll 1$ it is now easy to plot $\beta_e z_1$ versus $Z(=\beta z - \omega t)$, the normalized axial distance. This is done in Fig. 8b. Observe in this connection that $Z = Z_o + \beta z_1$ and that βz_1 is easily obtained from Fig. 8a. One sees that electron overtaking is just about to occur at $Z = \pi/2$ when R = 1.0. The dashed curve in
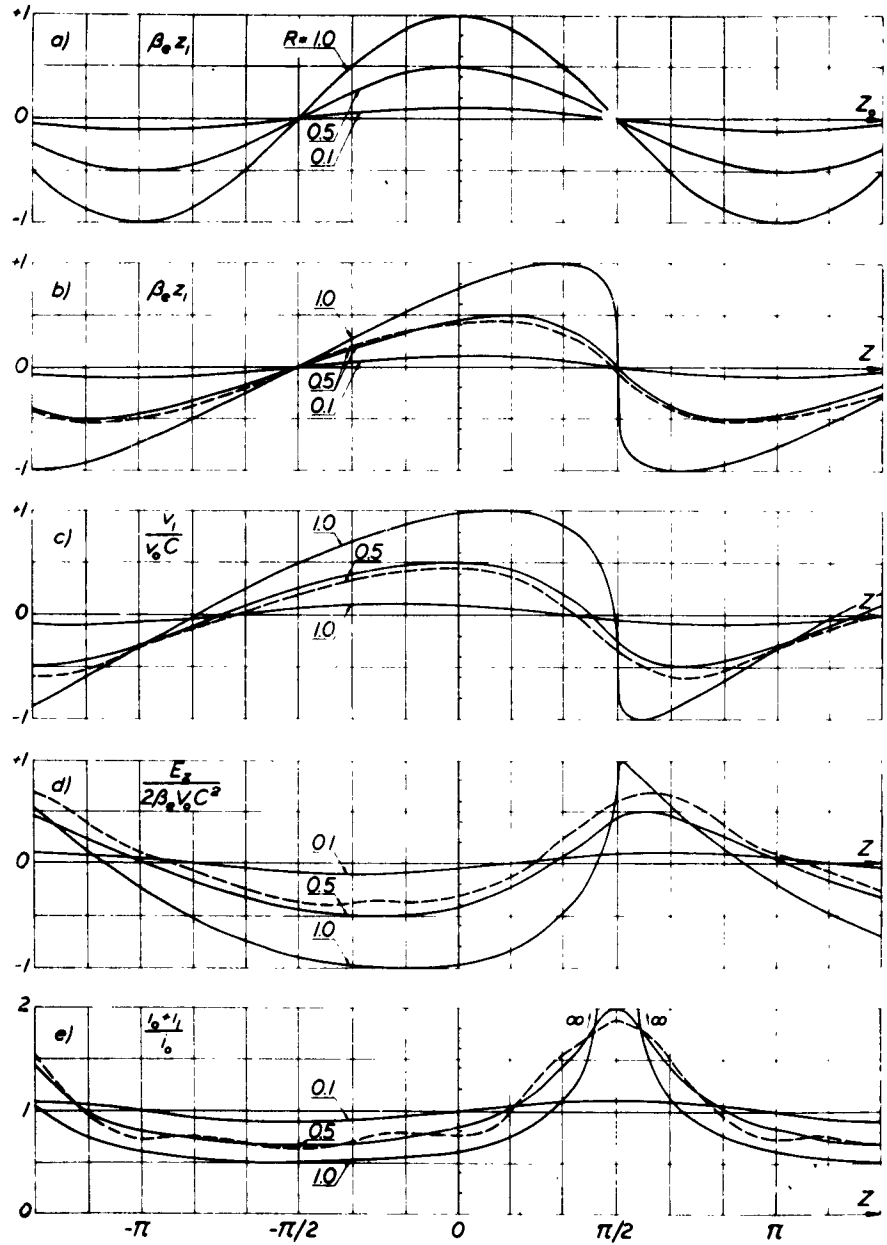
Fig. 8. Depicting a) the displacement (z_1) versus Z_0 and
b) the displacement (z_1), c) the excess velocity (v_1),
d) the axial electric field strength (E_z) and e) the total
electron current density $(i = i_0 + i_1)$ versus Z for three
values of the amplitude parameter R. The dashed curves, all
for R = 0.5, are taken from Fig. 4 for comparison.

Fig. 8b is taken from Fig. 4a for R = 0.5. It is obvious that the agreement between the results obtained with the successive approximation method on the one hand and with the dominant term[*] method on the other is good. The latter method is elegant and time saving but the price one pays is the neglect of information about phase shift and saturation.

If we write Equation (67) in the form

$$z_1 = A\,Re[e^{j(\omega t - \gamma z_o)}] = A\,Re[e^{j(\omega t - \gamma z + \gamma z_1)}] \tag{70}$$

we find by the use of Equation (1) that

$$\frac{v_1}{v_o C} = R\,Re[(x - jy)e^{jZ_o}] \tag{71}$$

In the case b = 0 we now find with $x = \frac{\sqrt{3}}{2}$ and $y = -\frac{1}{2}$ that

$$\frac{v_1}{v_o C} = R\cos(Z_o + 30^\circ) \tag{72}$$

Equation (72) has been plotted versus Z in Fig. 8c while graphs of Equation (71) are shown in Fig. 9a, b. From these curves one can see how the dc velocity v_o (i.e. the parameter b) influences the shape of the non-linear ac velocity v_1. Especially in Fig. 9a one notices that the faster the beam the more the velocity wave falls back in phase.

By the use of Equation (6) one obtains from Equation (70)

$$\frac{E_z}{2V_o \beta_e C^2} = R\,Re[(jx + y)^2 e^{jZ_o}] \tag{73}$$

---

[*] It should be remarked that Equation (67) is not yet of the form of Equation (61). We will return to this question and compute the dominant terms in the next Chapter.
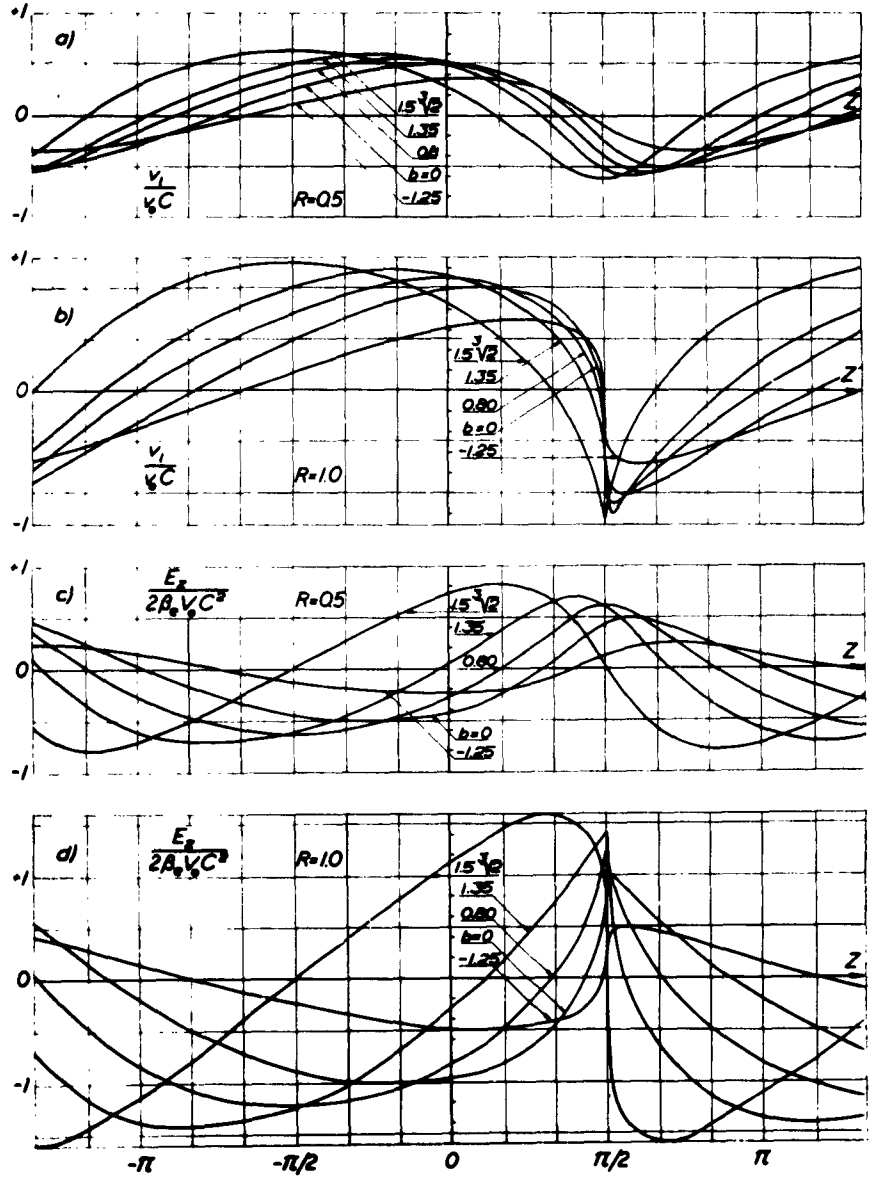
Fig. 9. Depicting a), b) the excess velocity (v_1) and c), d) the axial electric field (E_z) versus Z for different values of R and b.

For b = 0 this yields

$$\frac{E_z}{2V_0 \beta_e c^2} = R \sin(Z_0 - 30°) \qquad (74)$$

Equation (74) has been plotted in Fig. 8d and Equation (73) in Fig. 9c,d. Observe that the electric field wave moves back in phase when the dc velocity (i.e. the parameter b) increases.

By the use of Equation (4) one finally obtains from Equation (70)

$$\frac{i_0 + i_1}{i_0} = \frac{1}{1 - R \sin Z_0} \qquad (75)$$

Equation (75) is plotted versus Z in Fig. 8e. As in Fig. 4d one observes that a strong ac current bunch is built up around Z = $\pi/2$. From Fig. 9c,d one sees that choosing b = 0.8 would center the retarding ac field around the bunch. Thus one expects to obtain good efficiencies if the dc beam velocity is given a value somewhat higher than that corresponding to the maximum small signal gain. The experimental observations shown in Fig. 7 are in perfect agreement with this prediction.

In Fig. 10, finally, Equation (72) has been plotted for R = 2 and compared with Cutler's [8] measurement of the ac velocity in an experimental TWT. In the situation shown the drive is about 6 dB above overtaking and our theory is no longer valid. Nevertheless, the general shape of the two curves is essentially the same. However, it is beyond the scope of the present report to attempt to extrapolate the theory to the range beyond overtaking. It is interesting to note that overtaking occurs about 14 dB (6 dB + 8 dB) below saturation leaving a large and important drive range untouched by analytical theories.
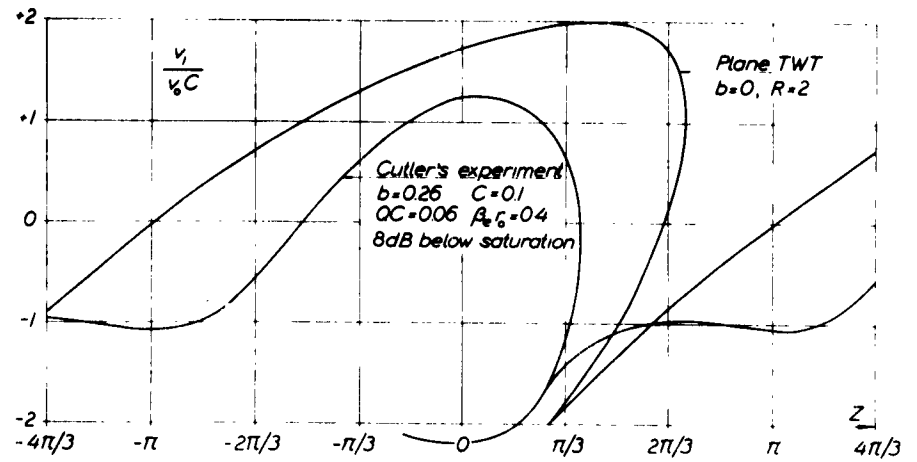
Fig. 10. The ac velocity versus phase, a comparison between Cutler's experiment and our Equation (72) in the range beyond overtaking.

VII. CALCULATION OF THE DOMINANT TERMS

In Chapter VI we reduced the exact fourth order wave equation (60) to an approximate second order wave equation (66). The exact solution (67) of the latter was then studied in detail. This solution made it possible to give simple expressions for the various wave quantities of interest. The purpose of the present chapter is to extract the dominant terms from the exact solution. The exact solution treated in Chapter VI is more convenient to handle than the series of dominant terms in order to obtain an overall physical picture of the non-linear phenomena. However, the exact solution is nothing more than a reasonably good approximation to the series of dominant terms. These will now be computed.

Use will be made of the theory previously developed for the plane klystron tube [12]. The exact wave equation of the plane klystron tube is

$$\frac{d^2 z_1}{dt^2} + \omega_p^2 z_1 = 0 \tag{76}$$

If the undisturbed beam is modulated at the plane $z = 0$ in an ideal klystron gap producing an ac velocity $v_1^o \cos \omega t$, the proper exact solution is

$$\beta_e z_1 = \frac{v_1^o \beta_e}{v_o \beta_p} \sin \beta_p z_o \, \cos(\omega t - \beta_e z_o) \tag{77}$$

If $|\beta_p z_1| << \pi/2$ one can write Equation (77) in the approximate form

$$\beta_e z_1 \simeq \frac{v_1^o \beta_e}{v_o \beta_p} \sin \beta_p z \, \cos(\omega t - \beta_e z + \beta_e z_1) \tag{78}$$

Equation (68), on the other hand, can be written

$$\beta_e z_1 \simeq R \cos(\omega t - \beta z + \beta_e z_1) \tag{79}$$

In the following table we compare the corresponding equations concerned with the plane TWT (approximate) and with the plane klystron (exact) respectively.

Plane TWT	Plane klystron
$\dfrac{d^2 z_1}{dt^2} + (\omega - \gamma v_o)^2 z_1 = 0$ (66)	$\dfrac{d^2 z_1}{dt^2} + \omega_p^2 z_1 = 0$ (76)
$R = A\beta_e e^{\alpha z}; \qquad T = \omega t - \beta z$	$R = \dfrac{v_1^o \beta_e}{v_o \beta_p} \sin\beta_p z; \qquad T = \omega t - \beta_e z$
$\beta_e z_1 \simeq R \cos(T + \beta_e z_1)$ (79)	$\beta_e z_1 \simeq R \cos(T + \beta_e z_1)$ (78)
$\beta_e z_1 \simeq \displaystyle\sum_{n=1}^{\infty} \frac{2}{n} J_n(nR) \sin\left[n\left(T+\frac{\pi}{2}\right)\right] \simeq \displaystyle\sum_{n=1}^{\infty} \frac{R^n}{n!}\left(\frac{n}{2}\right)^{n-1} \sin\left[n\left(T+\frac{\pi}{2}\right)\right] =$ $= R\cos T - \frac{1}{2}R^2 \sin 2T - \frac{3}{8}R^3 \cos 3T + \frac{1}{3}R^4 \sin 4T + \frac{125}{384}R^5 \cos 5T - \ldots$ (80)	
$\dfrac{i_1}{i_o} = \dfrac{\partial(\beta_e z_1)}{\partial T} \simeq \displaystyle\sum_{n=1}^{\infty} \frac{(nR)^n}{n!}\left(\frac{1}{2}\right)^{n-1} \cos\left[n\left(T+\frac{\pi}{2}\right)\right] =$ $= -R\sin T - R^2 \cos 2T + \frac{9}{8}R^3 \sin 3T + \frac{4}{3}R^4 \cos 4T - \frac{625}{384}R^5 \sin 5T \ldots$ (81)	

The last equations in the table are valid for both the TWT and the klystron, provided the proper expressions are used for R and T. These are listed in the table.

The series expansion for $\beta_e z_1$ [Eq. (80)] is obtained by the use of Fourier series, which is applied to the formally identical Equations (78) and (79), also listed in the table. The inversion procedure is given in detail in Reference [12],

Equations (37) through (45), and will not be repeated here.
It should be pointed out that the formula

$$Jn(nR) \simeq \frac{1}{n!}(\frac{nR}{2})^n$$

valid when $R^2 \ll 1$, has been used in Equation (80). Further-
more, a simple convergence test immediately gives the result
that the series

$$\sum_{n=1}^{\infty} \frac{R^n}{n!}(\frac{n}{2})^{n-1}$$

is convergent when $R < 2/e$. This, of course, does not tell
us very much about the precise range of validity of Equa-
tion (80), since the phase shift and saturation terms have
been ignored anyway.

If we introduce the notation $Z(= \beta z - \omega t = -T)$ into Equa-
tion (80) we obtain

$$\beta_e z_1 = R \cos Z + \frac{1}{2} R^2 \sin 2Z - \frac{3}{8} R^3 \cos 3T - \frac{1}{3} R^4 \sin 4Z + \frac{125}{384} R^5 \cos 5T \ldots$$

$$(82)$$

The reader should now compare Equation (82) with Equation
(40) and identify the three dominant terms which are common
in these two equations. Equation (40) is limited to the
third order and thus contains the three lowest dominant
terms but also the significant phase shift and saturation
terms up to the third order. There is also a second order
dc term in Equation (40). Equation (80), on the other hand,
gives the dominant terms of all orders but misses the phase
shift and saturation terms completely.

From Equation (81) and Fig. 11 one concludes that,
unless $R \ll 1$, the amplitudes of the harmonic frequency
waves are large. This, of course, is expected on account
of the development of the strong and narrow electron current
bunch around the phase $Z = \pi/2$ when R is large enough. At
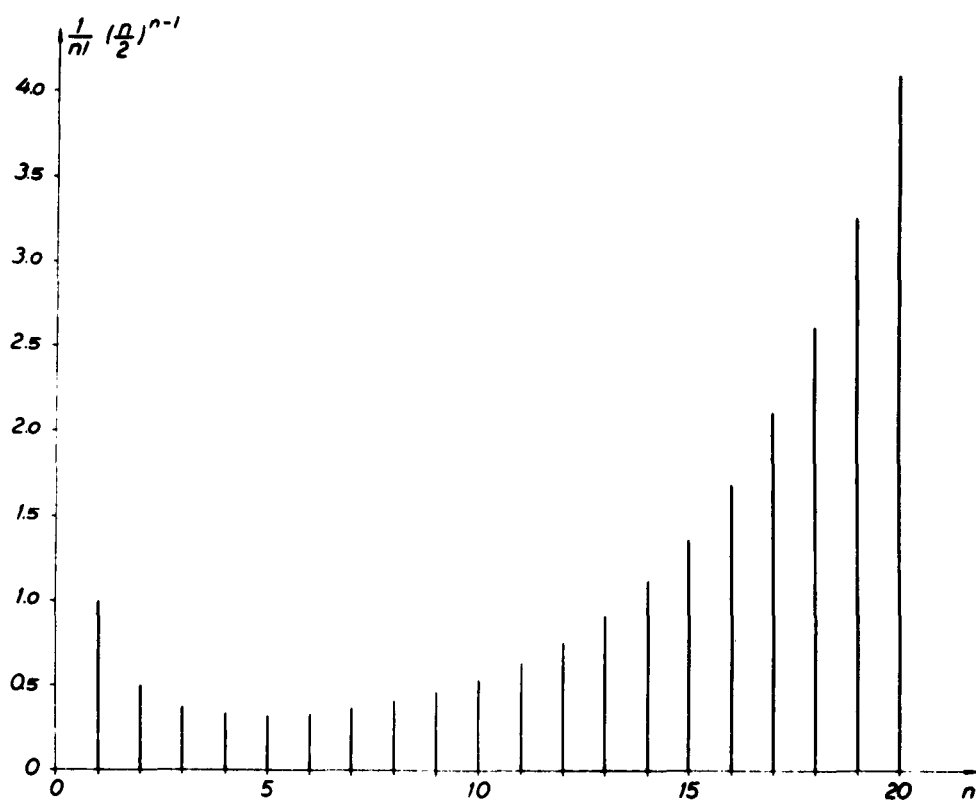the same time the coupling between the electron beam and

$$\frac{1}{n!}\left(\frac{n}{2}\right)^{n-1}$$

Fig. 11. Depicting the coefficients appearing in Equation (80) versus n.
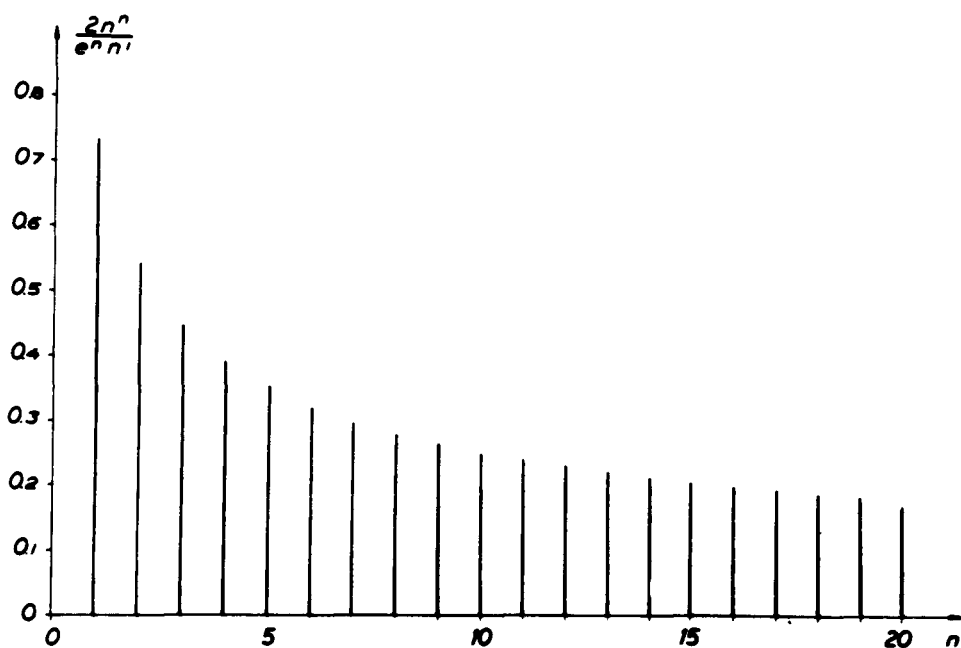
$$\frac{2n^{n}}{e^{n}n!}$$

Fig. 12. The amplitudes of the normalized ac harmonic electron current waves, $\frac{(nR)^{n}}{n!}\left(\frac{1}{2}\right)^{n-1}$ at the limit of convergence of Eq. (81), i.e. for R = 2/e.

the slow wave medium is, for natural reasons, strong even
at harmonic frequencies. In a real traveling wave tube one
should expect a much less pronounced generation of harmo-
nic frequencies due to weak coupling.

VIII. CONCLUDING REMARKS

This report is concerned with a purely analytical non-linear theory of a plane traveling wave tube (TWT) model. The model is described in Chapter I. It consists of an infinitely wide confined, single-velocity electron plasma stream of zero temperature which interpenetrates an anisotropically conducting homogeneous medium. In the absence of the electron stream the medium supports plane electromagnetic slow TM (transverse-magnetic) waves. Thus in our model the medium plays the same role as the slow wave structure in a real TWT. The reason for choosing a plane model is that by so doing one avoids the theoretical complications associated with the radial boundary conditions. This means, of course, that the present theory does not describe such TWT effects which are caused by the radial finiteness of the beam and/or the slow wave structure. Instead the model allows one to study in the simplest conceivable manner the basic physical effects associated with the TWT amplification mechanism itself.

In Chapter II the exact non-linear wave equation [Eq. (10)] for the plane TWT has been derived by the use of Maxwell's equations. It is assumed that the electron velocity is a one-valued function of distance, i.e. electron overtaking is not allowed. This restriction considerably simplifies the analysis but at the same time it limits the theory to drive levels well below saturation.

The results of a linearized theory of the plane TWT are briefly discussed in Chapter III. For a detailed linear analysis of the device the reader is referred to References [1a,b]. In Chapter IV the method of successive approximations is applied in order to obtain a non-linear third order solution. Some of the results are demonstrated in Fig. 4, which shows how the sinusoidal form of the various wave quantities is distorted by non-linear processes when the wave amplitudes become large.

The non-linear effects appearing in the amplitude and

phase of the fundamental frequency wave are studied in Chapter V (see Fig. 5 and 6). A number of new and rather interesting results are derived and discussed. Some of these have been checked experimentally by the use of a commercial TWT. The qualitative agreement between the theory and the experiments is good (Fig. 7).

Chapters VI and VII are concerned with another method (method of dominant terms) to deal with the non-linear fourth order wave equation [Eq. (10)]. This method ignores some of the important non-linear effects (saturation, phase shift). On the other hand it quickly and elegantly leads to simple implicit expressions for the wave quantities. Fig. 8, depicting these expressions, shows that the agreement between the two methods (successive approximations and dominant terms) is close. In Chapter VII the dominant term type approximate theory of the plane TWT is shown to be formally identical to the exact theory of the plane klystron tube, which has been treated earlier [12]. By the use of the plane klystron theory the dominant terms, i.e. the leading terms at each harmonic frequency $n\omega$, $n = 1, 2, 3, \ldots \infty$, are computed for the plane TWT.

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ACKNOWLEDGMENTS

The author gratefully acknowledges the assistance given
by Mr. M.Fischer, who checked the results and prepared most
of the graphs, and by Mr. J.-I.Lindström and Mr. T.Srinivasan,
who performed the measurements.

The work has been made possible through the support of
the Rome Air Development Center, Air Force Systems Command,
through the European Office, Aerospace Research, United States
Air Force.

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RES. LAB. OF ELECTRONICS   AF 61(052)-594
CHALMERS UNIV. OF TECH.   TN-1
GOTHENBURG, SWEDEN   ELECTRONICS
RESEARCH REPORT NO 37
1 March 1963

ON THE NON-LINEAR THEORY OF THE PLANE TRAVELING WAVE TUBE

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ABSTRACT: The report deals with an analytical non-linear theory of a planar model of the traveling wave tube. The effect of non-linearities on fundamental frequency amplitude saturation and phase shift, the generation of harmonics and the distortion of waveforms is discussed.